On rationally integrable planar projective billiards

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Consider a billiard in a strictly convex planar domain bounded by a smooth curve, with the classical law of line reflection: the angle of incidence is equal to the angle of reflection. A *caustic* of a billiard is a curve C whose tangent lines are reflected by the billiard to its tangent lines (e.g., a confocal ellipse in an elliptic billiard). The famous **Birkhoff Conjecture** deals with **Birkhoff integrable** billiards, that is, *admitting a foliation by closed caustics in a neighborhood of the boundary from the inner side, with boundary being a leaf.* It affirms that the *only Birkhoff integrable billiards are ellipses.* The Birkhoff Conjecture was studied by many mathematicians, including H.Poritsky, M.Bialy, A.Mironov, V.Kaloshin, A.Sorrentino.

Sergei Tabachnikov suggested a generalization of the Birkhoff Conjecture to the projective billiards. His conjecture would imply all the versions of the Birkhoff Conjecture for surfaces of non-zero constant curvature and for outer planar billiards. Tabachnikov's Conjecture is stated in dual terms. Namely, consider a planar strictly convex closed curve C and a foliation by closed curves of its neighborhood on the concave side, with C being its leaf. For every point $P \in C$ let L_P denote the projective line tangent to C at P. Consider the germ at P of involution $L_P \to L_P$ permuting its intersection points with each individual leaf of the foliation. Suppose that *for every $P \in C$ the latter involution is a projective transformation* of the tangent line. **Tabachnikov's Conjecture** affirms that under these assumptions *the curve C is an ellipse, and the foliation in question is a pencil of conics.*

In the talk we present a proof of the *positive answer* under the additional assumption that the foliation admits a *rational first integral.*

We also prove a *local version:* in the case, when C is a C^4-germ of planar curve and the germ of foliation admits a rational first integral. We show that the curve C is conic, but the leaves of the foliation may be higher degree algebraic curves. We give a complete classification of germs of foliations satisfying the conditions of local Tabachnikov's Conjecture admitting rational first integrals, up to projective transformation. Their list includes two infinite series of exotic examples, with higher degree leaves, and two pairs of more exceptional examples, with leaves of degrees four and six.