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Convolution equations on the Lie group $G = (-1, 1)$ and their applications

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Abstract

For each Lie group G equiped with a manifold and the group operation (multiplication or sum $x \circ y$, inverse element, neutral element) there exists the unique left or right invariant Haar measure $d_G x$, homeomorphism $c(x) : G \to \hat{G}$ to an unitary group $\hat{G}$, called the representation of the Lie group G and, finally, perhaps the most important, the Fourier transformation $\mathcal{F}_G$, which is an isomorphism of the Lebesgue-Hilbert spaces $\mathcal{F}_G : \mathbb{L}_2(G) \to \mathbb{L}_2(\hat{G})$, with the inverse transformation $\mathcal{F}_G^{-1}$. Most interesting in applications are convolution equations on different Lie groups

$$W_a^0 \varphi(x) := \mathcal{F}_G^{-1} a \mathcal{F}_G \varphi(x) = c\varphi(x) + \int_G k(x \circ y^{-1}) \varphi(y) d_G y = f(x), \qquad x \in G. \tag{1}$$

where the Fourier transform of the kernel $a(\xi) = c + (\mathcal{F}_G k)(\xi)$, $\xi \in \hat{G}$ is called the symbol. Under certain conditions equation (1) has a unique solution $\varphi(x) = (W^0_{a^{-1}} f)(x)$. To such equations belong celebrated classical convolution equations of Wiener-Hopf on the axes $\mathbb{R} = (-\infty, \infty)$, of Mellin equations on the half axes $\mathbb{R}^+ = (0, \infty)$ and Töplitz equations on the grid of integers $\mathbb{Z} = \{0, \pm1, \pm2, \cdots\}$. These equations have ample of applications in problems of Mathematical Physics, Probability theory, Elasticity theory etc. I will speak about a Lie algebra-interval $G = (-1, 1)$, which is equiped with the group operation $x \circ y := (x + y)(1 + xy)^{-1}$, $x, y \in G$. The invariant Haar measure is $d_G x := (1 - x^2)^{-1} dx$ and the Fourier transformation

$$(\mathcal{F}_{\mathcal{G}} v)(\xi) := \int_{-1}^{1} \left(\frac{1-y}{1+y}\right)^{i\xi} \frac{v(y)dy}{1-y^2} = \int_{-1}^{1} \left(\frac{1-y}{1+y}\right)^{i\xi} v(y) d_G y, \qquad \xi \in \mathbb{R}. \tag{2}$$

These tools allow to solve exactly convolution integro-equations on this group

$$\sum_{k=0}^{n} \left[a_k \mathfrak{D}^k u(t) - b_k \int_{-1}^{1} \left(\frac{1-\tau^2}{1-t^2}\right)^{\alpha_k} \frac{\mathfrak{D}^k u(\tau) d\tau}{\tau - t} \right] = f(t), \qquad a_k, b_k, \alpha_k \in \mathbb{C}, \quad t \in \mathcal{J}, \tag{3}$$

where $\mathfrak{D}u(x) := -(1 - x^2)\dfrac{d}{dx} u(x)$ is the natural derivative of functions on the group G (the generator of the Lie algebra) and $(\mathcal{F}_G \mathfrak{D}) = -2i\xi$, $\xi \in \mathbb{R}$. It turned out that to the class of convolution equations (3) belong the following celebrated airfoil (Prandtl) equation with important applications

$$\boldsymbol{P}u(x) = \frac{c_0 u(x)}{1 - x^2} + \frac{c_1}{\pi i} \int_{-1}^{1} \frac{u'(y) dy}{y - x} = f(x), \qquad x \in \mathcal{J} \tag{4}$$

Also singular Tricomi and Lavrentjev-Bitsadze equation, which emerge in solving partial differential equations of mixed type. Moreover, Laplace-Beltrami equation on the unit sphere in $\mathbb{S}^2 \subset \mathbb{R}^3$ is also a G-convolution operator with a parameter. In conclusion we touch recent results obtained in collaboration with Duván Cardona, Arne Hendrickx & Michael Ruzhansky (Ghent University, Belgium) concerning Global pseudo-differential operators on the Lie group (cube) $G = (-1, 1)^n$.

Mixed type boundary value problems for the Helmholtz equation in a model 2D double angular domain

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Abstract

The purpose of the present research is to investigate model mixed boundary value problems for the Helmholtz equation in a model 2D double angular domains $\Omega_\alpha \subset \mathbb{R}^2$ of magnitude α and $\Omega_{-\beta} \subset \mathbb{R}^2$ of magnitude β, where $\alpha, \beta > 0$. The BVP is considered in a non-classical setting, when solutions are sought in the Bessel potential spaces $\mathbb{H}^s_p(\Omega_\alpha)$, $s > 1/p$, $1 < p < \infty$. The problems are investigated using the potential method by reducing them to an equivalent boundary integral equation (BIE) in the Sobolev-Slobodečkii space on a semi-infinite axes $\mathbb{W}^{s-1/p}_p(\mathbb{R}^+)$, which is of Mellin convolution type. By applying the recent results on Mellin convolution equations in Bessel potential spaces obtained by V. Didenko & R. Duduchava in [2], explicit conditions of the unique solvability of this BIE in the Sobolev-Slobodečkii $\mathbb{W}^r_p(\mathbb{R}^+)$ and Bessel potential $\mathbb{H}^r_p(\mathbb{R}^+)$ spaces for arbitrary r are found and used to write explicit conditions for the Fredholm property and unique solvability of the initial model BVPs for the Helmholtz equation in the above mentioned non-classical setting. These results, together with the results of the paper [3], where the model Dirichlet and Neumann BVPs in angular domains are investigated, will be used in a forthcoming paper to derive unique solvability criteria for mixed boundary value problems for the Laplace-Beltrami equation on a hypersurface $\mathcal{C} \subset \mathbb{R}^3$ with the Lipschitz boundary $\Gamma = \partial\mathcal{C}$.

References

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