A global Riemann-Hilbert problem for two-dimensional inverse scattering at xed energy

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Imagine that one instructs a monkey to write formal mathematical expressions and we ask it to propose an equation for us to solve. The truth of life is that mathematicians, almost for sure, will not know how to solve this equation explicitly, i.e. creating a formula as we solve quadratic equations in high school. So one can consider Mathematics as an art of tricks to solve a vanishing percentage of possible problems a monkey can propose. Another side of this truth is the following: if a problem is important enough, several generations of mathematicians are able to find a sequence of tricks which lead to a solution. Very rarely the miracle happens that a new trick allows to solve a pack of problems given by a monkey named Nature.

It happened exactly 50 years ago in 1967 when a group of mathematicians discovered that the KdV equation (1D water equation, simplest among nonlinear equations) can be reduced to a couple of linear equations [1]. In the following years it was discovered that the same procedure is applicable for many other important 1D and 2D equations. For example, equations that mode nonlinear propagation of light in an optic fiber, propagation of water waves, Heisenberg ferromagnetic model. The early stage of this progress is connected with names of Lax, Faddeev, Zakharov and Shabat. Although, in general the idea of this method has been clearly absorbed by scientific community, the proposed technique is not a set of computer applicable methods, but contains a piece of art: How can the problem be reduced to a Riemann-Hilbert factorization problem?

During 1980s attempts to solve some 2D equation (Novikov-Veselov, Davey-Stewartson II, Kadomtsev–Petviashvili II and Ishimori) met a common problem: the possible appearance of "curves" of special points, exactly those points which in 1D correspond to travelling waves: solitons. In the 1D case such solitons form a set of isolated points and do not represent a serious technical problem, but in 2D they can appear as a curve and analogues of its successful treatment did not exist.

In 1991 the monograph by Ablowitz and Clarkson [2] on nonlinear evolution equations joined these problems together and stated the overcoming of this common difficulty as an open problem. The art of solving nonlinear equations equations has been enriched in 2016 with appearance of the article [3] where authors proposed a method of reducing to the Riemann-Hilbert problem in the possible presence of "soliton curves". Let us give a brief technical insight into the developed method. In 1D scattering problem the solution allows an analytic continuation into the complex plane of non-physical energies. The quotient of the limiting values equals to the scattering amplitude and therefore the Riemann-Hilbert problem is a natural tool to reconstruct the scattering solution via the known scattering amplitude. In 2D scattering problem there does not exist an appropriate analytic continuation into the plane of non-physical energies, instead of that the quotient of the scattering solution and its partial derivative (dbar) equals to the scattering amplitude. In this case the inverse procedure is given by a superposition of the dbar-problem in the whole complex plane and the non-local Riemann-Hilbert problem at the real line. This beautiful idealistic situation holds only in the case of absence of “soliton” energies. Authors of [3] developed an alternative: if one chose a finite domain D such that all possible “soliton” points or curves are inside D, then the inverse procedure can be reduced to a superposition of the dbar-problem outside D and the non-local Riemann-Hilbert problem at the boundary of D.

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